

WHICH Q -ANALOGUE OF THE SQUEEZED OSCILLATOR?

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1 Introduction and Content

The noise (variance squared) of a component of the electromagnetic field - considered as a quantum oscillator - in the vacuum is equal to one half, in appropriate units (taking Planck's constant and the mass and frequency of the oscillator all equal to 1). A practical definition of a squeezed state is one for which the noise is less than the vacuum value - and the amount of squeezing is determined by the appropriate ratio. Thus the usual coherent (Glauber) states are not squeezed, as they produce the same variance as the vacuum. However, it is not difficult to define states analogous to coherent states which do have this noise-reducing effect. In fact, they are coherent states in the more general group sense but with respect to groups other than the Heisenberg-Weyl Group which defines the Glauber states. The original, conventional squeezed state in quantum optics is that associated with the group $SU(1,1)$. Just as the annihilation operator a of a single photon mode (and its hermitian conjugate a^\dagger , the creation operator) generates the Heisenberg-Weyl algebra, so the pair-photon operator a^2 and its conjugate generates the algebra of the group $SU(1,1)$. Another viewpoint, more productive from the calculational stance, is to note that the automorphism group of the Heisenberg-Weyl algebra is $SU(1,1)$. Needless to say, each of these viewpoints generalizes differently to the quantum group context. In this talk we shall discuss both. The structure of the talk is as follows:

- Conventional Coherent and Squeezed States
- Eigenstate Definitions
- Exponential Definitions
- Algebra (Group) Definitions
- Automorphism Group Definition
- Example: Signal-to-Noise Ratio
- q -Coherent and q -Squeezed States
- M and P q -bosons
- Eigenstate Definitions

- Exponential Definitions
- Algebra (q -Group) Definitions
- Example: Signal-to-Noise Ratio
- Automorphism q -Group Definition

2 Conventional Coherent and Squeezed States

The elementary treatment of (a single frequency) of the quantized electromagnetic field leads to the identification of its components

$$E \sim x, \quad B \sim p, \quad [x, p] = i$$

in suitably chosen units. We may introduce boson operators b, b^\dagger by

$$x = (b + b^\dagger)/\sqrt{2} \quad p = (b - b^\dagger)/\sqrt{2}i$$

which then satisfy the Heisenberg-Weyl Algebra

$$\begin{aligned} [b, b^\dagger] &= 1 \\ [N, b^\dagger] &= b^\dagger \end{aligned} \quad (1)$$

where $N = b^\dagger b$. The interpretation of these operators is that they annihilate (*resp.* create) photons; the *vacuum state* $|0\rangle$ satisfies

$$b|0\rangle = 0.$$

The quantum noise of the x -component (E -field) in the vacuum state is given by

$$(\Delta x)^2 \equiv \langle x^2 \rangle - \langle x \rangle^2 = 1/2$$

with a similar result for the B -component. The vacuum signal ($\langle x \rangle^2$) vanishes.

The conventional *coherent* states (Glauber [1] states) are defined as eigenstates of the operator b ,

$$b|\lambda\rangle = \lambda|\lambda\rangle. \quad (2)$$

For these states one readily evaluates

$$(\Delta x)^2 = 1/2 \quad \langle x \rangle^2 = (\lambda + \bar{\lambda})^2/2.$$

An alternative, suggestive definition of the coherent states which readily lends itself to generalization, is that they are obtained by the action of the realizations of the group corresponding to the Heisenberg-Weyl Algebra generated by $\{b, b^\dagger, 1\}$ on the vacuum, thus;

$$|\lambda\rangle = \exp(\lambda b^\dagger)|0\rangle. \quad (3)$$

It is an important practical problem to maximize the signal-to-noise ratio ρ for radiation states; here we are of course only considering the quantum noise. What we see from the preceding is that ρ vanishes for the vacuum; it attains the value $4N_s$ for a coherent state, taking (real) $\lambda^2 = N_s$, where N_s is the number of photons in the signal.

In a classic paper, Yuen [2] showed that for any radiation field the maximum signal-to-quantum noise ratio ρ for fixed energy has the value $\rho_{max} = 4N_s(N_s + 1)$, where N_s gives the upper limit on the number of photons in the signal (effectively a maximum power per unit frequency constraint). He further showed that this value is attained by the *squeezed* states [3], two-photon coherent light states generated as eigenstates of the operator $\mu b + \nu b^\dagger$ where $|\mu|^2 - |\nu|^2 = 1$. The only mathematical input to this result consists of the canonical commutation relations Equation (1). The term "squeezed" derives from the fact that in these states the quantum dispersion may attain values below the vacuum (or coherent) state value of $1/2$. Such states have been produced experimentally. These squeezed states may also be defined by the action on the vacuum (more generally, on Glauber coherent states) of the group corresponding to the algebra generated by

$$\{b^2, (b^\dagger)^2, (bb^\dagger + b^\dagger b)\}. \quad (4)$$

Thus a typical squeezed state (up to normalization) may be written

$$|\xi, z\rangle = \exp\left(\frac{1}{2}\xi(b^\dagger)^2\right) \exp(zb^\dagger)|0\rangle. \quad (5)$$

The state $|\xi, z\rangle$ is an eigenstate of $(b - \xi b^\dagger)$ with eigenvalue z , in agreement with the definition of squeezed state above ($\mu = 1, \nu = -\xi$ and for convergence we require that $|\xi| \leq 1$.) The operators in Equation (4) satisfy the commutation relations of $SU(1, 1)$

$$\begin{aligned} [K_+, K_-] &= -2K_0 \\ [K_0, K_\pm] &= \pm K_\pm. \end{aligned} \quad (6)$$

An alternative definition which results in states exhibiting squeezing is to define them as (normalized) eigenstates of the of the lowering operator $K_- \equiv b^2$. These states have the form

$$\begin{aligned} |\xi_1\rangle &= \sum_{i=0}^{\infty} \frac{\xi_1^i}{\sqrt{2^i i!}} |2i\rangle \\ |\xi_2\rangle &= \sum_{i=0}^{\infty} \frac{\xi_2^i}{\sqrt{(2i+1)!}} |2i+1\rangle. \end{aligned} \quad (7)$$

An appropriate sum of these squeezed states recovers a Glauber coherent state.

A more basic definition of squeezed states arises from the observation that the automorphism group of the H-W algebra is $SU(1, 1)$; thus a unitary transformation U on b gives

$$b \mapsto U b U^\dagger = \mu b + \nu b^\dagger \quad (8)$$

where $|\mu|^2 - |\nu|^2 = 1$. The conventional squeezed state is then defined, exactly as above, as an eigenstate of the transformed bose destruction operator

$$(\mu b + \nu b^\dagger)|\xi\rangle = \xi|\xi\rangle. \quad (9)$$

More generally, a conventional squeezed state is defined as the action of the unitary operator $U(\mu, \nu)$ on a coherent state $|z\rangle = D(z)|0\rangle$, thus:

$$|\xi\rangle = U(\mu, \nu)D(z)|0\rangle = U(\mu, \nu)|z\rangle \quad (10)$$

This definition is not only elegant but, by applying the inverse transformation, enables calculations in squeezed states to be made as readily as in the coherent states.

For example; using

$$U(\mu, \nu)^{-1} b U(\mu, \nu) = \bar{\mu}b - \nu b^\dagger$$

one may readily evaluate the dispersion of x in the squeezed state $|\xi\rangle$ to be

$$(\Delta x)^2 = \frac{1}{2}|\mu - \nu|^2$$

and the signal to be

$$\langle x \rangle^2 = \{(\mu - \nu)\bar{z} + \overline{(\mu - \nu)z}\}^2.$$

For real values of the parameters, the maximum of the signal-to-quantum noise ratio

$$\rho \equiv \langle x \rangle^2 / (\Delta x)^2 = 4z^2$$

may readily be seen to be attained at $\rho_{max} = 4N_s(N_s + 1)$ as cited above [2].

3 q -Coherent and q -Squeezed States

A deformation a_M of the standard boson operator b was introduced some years ago by Arik and Coon [4]. Their deformed bosons satisfy

$$a_M a_M^\dagger - q a_M^\dagger a_M = 1. \quad (11)$$

More recently, the deformed q -boson operator a_P satisfying the Quantum Heisenberg-Weyl Algebra ($H - W_q$ Algebra)

$$\begin{aligned} a_P a_P^\dagger - q a_P^\dagger a_P &= q^{-N} \\ [N, a_P^\dagger] &= a_P^\dagger \end{aligned} \quad (12)$$

has been introduced [5, 6]. (I have used the subscript M to denote the relation to the mathematician's classical q -analysis, a study which goes back at least as far as Gauss, in contrast to the more recently introduced physicist's form, subscript P. The second equation of (12) is satisfied by both forms. There is no need to subscript the operator N for the reason given below.)

In principle, either Equation (2) or Equation (3) can be used as a starting point for an eigenstate definition of q -coherent states for both types of deformed bosons. It is easily shown that an attempt to use Equation (3) does not lead to a normalizable state (for $q \neq 1$) in either case. Starting from Equation (2), q -coherent states for the deformed boson operator of Arik and Coon were constructed

by these authors [4]; the same equation was used [6] for the q -bosons defined in Equation (12). Both forms of q -boson lead to the q -coherent state

$$|\beta \rangle_q = \mathcal{N}^{-1} \exp_q(\beta a^\dagger) |0 \rangle \quad (13)$$

where $a = a_M$ or a_P and

$$\mathcal{N}^2 = \exp_q(|\beta|^2). \quad (14)$$

The q -exponential is defined in both cases by

$$\exp_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!}. \quad (15)$$

The symbol $[r]_q!$ is defined by $[r]_q! = [r]_q[r-1]_q[r-2]_q \cdots [1]_q$ where, in the case of Equation (11), we define

$$[x]_q = (q^x - 1)/(q - 1) \quad (16)$$

and in the case of Equation (12), we define

$$[x]_q = (q^x - q^{-x})/(q - q^{-1}). \quad (17)$$

Equation (11) gives rise to the classical form of the q -exponential usually written as $E_q(x)$, which converges for $|q| > 1$, or for $|x| < |\frac{1}{1-q}|$ when $|q| < 1$. The form of q -exponential corresponding to Equation (12) is convergent for all x and q . In both cases, $\lim_{q \rightarrow 1} \exp_q(x) = \exp(x)$, and when $q = 1$ the q -boson operators reduce to standard boson operators.

The q -bosons are related to the conventional bosons b as follows:

$$a = b \sqrt{\frac{[N]_q}{N}} \quad (18)$$

where $N = b^\dagger b$, using the appropriate definition of $[N]_q$ for “mathematical” bosons Equation (16) [7] or “physical” bosons Equation (17) [8].

The q -coherent states defined above do not give rise to (time-independent) squeezing, just as in the case of the conventional coherent states. In fact, it may be shown [7] that the term which gives rise to squeezing is, in general,

$$\langle a^2 \rangle - \langle a \rangle^2$$

which is zero for eigenstates of a . However, Buzek [9] has shown that there is time-dependent squeezing, by choice of a suitable analogue of the usual Hamiltonian; and this has also been found by Celeghini, Rasetti and Vitiello [10].

It is not immediately clear how conventional squeezed states can be generalized to the quantum group context. The most direct approach is to use a q -boson realization of the analogous $su_q(1, 1)$ algebra; one may then attempt to define the analogous q -squeezed states by the exponential action

of the resulting operators on the vacuum (or on the q -coherent states). A realization of $su_q(1, 1)$ given by Kulish and Damaskinsky [8] is

$$K_+ = p(a^\dagger)^2 \quad K_- = p(a)^2 \quad K_0 = \frac{1}{2}(n + \frac{1}{2}) \quad (19)$$

with $p = (q + q^{-1})^{-1}$ and $[K_+, K_-] = -[2K_0]_{q^2}$. However, the exponential action of the operators of this algebra fail to give a normalizable state not only for the conventional exponential (which was to be expected) but also for the q -exponential $\exp_q(x)$ defined above (and also for $\exp_{q^2}(x)$ which one would have thought to be the appropriate function here).

The eigenstates of K_- corresponding to Equations (7), obtained by substituting the "box" factorials for the conventional ones, give normalizable states [11].

We may alternatively carry over the definition

$$(a - \xi a^\dagger)|\xi, z \rangle = z|\xi, z \rangle \quad (20)$$

to the q -boson case. For the choice $z = 0$ we obtain

$$|\xi, z \rangle = \mathcal{N}^{-1} \sum_{i=0}^{\infty} \xi^i \sqrt{\frac{[2i-1]_q!!}{[2i]_q!!}} |2i \rangle \quad (21)$$

with normalization

$$\mathcal{N}^2 = \sum_{i=0}^{\infty} |\xi|^{2i} \frac{[2i-1]_q!!}{[2i]_q!!}. \quad (22)$$

The symbol $[r]_q!!$ has the expected meaning $[r]_q!! = [r]_q[r-2]_q[r-4]_q \cdots$ and the first term in (22) is 1. The squeezing properties of states defined in this way, for various values of the parameters ξ and q were calculated in [11].

A more basic definition of squeezed states in the quantum group case arises from generalizing the automorphism group property given in the previous section for the conventional case, Equation (8) and Equation (9). One may seek by analogy to define q -squeezed states in terms of the automorphism quantum group of the quantum Heisenberg-Weyl algebra of q -bosons. Consider the quantum plane *à la Manin* generated by two elements α and γ as defined by Woronowicz [12], satisfying the following commutation relations:

$$\begin{aligned} \alpha\gamma &= \mu\gamma\alpha \\ \alpha\gamma^* &= \mu\gamma^*\alpha \\ \gamma\gamma^* &= \gamma^*\gamma \\ \alpha^*\alpha - \gamma^*\gamma &= 1 \\ \alpha\alpha^* - \mu^2\gamma^*\gamma &= 1. \end{aligned} \quad (23)$$

We now introduce a conjugation $A \mapsto \hat{A}$ defined by its effect on

1. c-numbers $c \mapsto c^*$, (complex conjugation)
2. q -numbers (quantum plane) $\tilde{\alpha} = \alpha^*$ $\tilde{\alpha}^* = \alpha$ $\tilde{\gamma} = \mu\gamma^*$ $\tilde{\gamma}^* = \frac{1}{\mu}\gamma$

3. operators $\tilde{A} = q^{\frac{1}{2}(N^2-N)} A^\dagger q^{-\frac{1}{2}(N^2-N)}$ (q real).

Under this transformation, $\tilde{A} = A$, $\tilde{A}B = \tilde{B}\tilde{A}$; and the boson a satisfying

$$aa^\dagger - qa^\dagger a = q^{-N}$$

maps to \tilde{a} , with the pair a, \tilde{a} satisfying

$$a\tilde{a} - \mu\tilde{a}a = 1. \quad (24)$$

with $\mu = q^2$. The two-dimensional fundamental representation of $SU_\mu(1,1)$ is given by

$$u = \begin{bmatrix} \alpha & \mu\gamma^\star \\ \gamma & \alpha^\star \end{bmatrix} \quad (25)$$

and u satisfies $uJu = J$ where

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -\mu \end{bmatrix}$$

The transformation

$$[a, \tilde{a}] \mapsto [a, \tilde{a}]u \quad (26)$$

is an automorphism which preserves Equation (24). Squeezed states in the quantum group context may now be defined as the eigenstates of the transformed a , thus generalizing the results of [11].

Finally, we note that one may derive an analogue of Yuen's result [2] cited above on the optimal signal-to-Quantum Noise ratio q -photons [13]; the corresponding bound for q -photons may be shown to be

$$\rho_q = 4[N_s]_q[N_s + 1]_q / ([N_s + 1]_q - [N_s]_q)^2. \quad (27)$$

that is, for a radiation field in terms of photons satisfying the modified commutation relations of the quantum group version of the Heisenberg-Weyl Algebra. This ratio is always less than the value in the conventional case, attained for the $SU(1,1)$ squeezed states.

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